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A Typical Lower Bound for Odds Problem in Markov-dependent Trials (Stochastic Decision Analysis)

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A Typical Lower Bound for Odds Problem in Markov-dependent Trials ¹

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1 Introduction

We study a stopping problem for Markov-dependent trials of the odds problem. It may be described as follows. For a positive integer N , let X_1, X_2, \dots, X_N denote 0/1 random variables defined on a probability space (Ω, \mathcal{F}, P) . These 0/1 random variables appears according to non-homogenous Markov chain with the transition probability such that

$$\mathbf{P}_i = \begin{pmatrix} 1 - \beta_i & \beta_i \\ \alpha_i & 1 - \alpha_i \end{pmatrix}, \quad (1)$$

where $\beta_i := P(X_{i+1} = 1 | X_i = 0)$, $\alpha_i := P(X_{i+1} = 0 | X_i = 1)$, $\beta_0 := P(X_1 = 0)$ and $\alpha_0 := P(X_1 = 1) = 1 - \beta_0$. Each α_i and β_i are given. We assume $0 < \alpha_i, \beta_i < 1$ for all i . We observe these X_i 's sequentially and claim that the i th trial is a *success* if $X_i = 1$. We want to find the optimal stopping rule that maximize the probability of obtaining the last success (we call this event *win*) and the probability of win.

If $0 < \alpha_i + \beta_i < 1$ for all $i = 1, 2, \dots, N$ in the transition probability, then Hsiao and Yang [7] found the optimal rule, but it was not of odds form. Bruss [5] proved that the lower bound for odds problem in Bernoulli trials is $1/e$ for any sequence of success probabilities, $P(X_i = 1)$, $i = 1, 2, \dots, N$.

The main result of this paper is that the asymptotic lower bound of probability of win is also $1/e$ for any transition probability of Markov chain under a certain condition. I think it is wonderful!

2 Main result

Recall that Ano, Kakie and Miyoshi [3] proved that even though it is for Markov-dependent trials, the optimal stopping rule can be expressed as of odds form. Let

$$p_{ij} := \begin{cases} P(X_{i+1} = 1 | X_i = 1, X_{i+2} = 0) = (1 - \alpha_i)\alpha_{i+1}, & j = i + 1, \\ P(X_{i+1} = 1 | X_{j-1} = 0, X_{j+1} = 0) = \beta_{j-1}\alpha_j, & j > i + 1, \end{cases}$$

and $r_{ij} = p_{ij}/(1 - p_{ij})$. This is our key setting inspired by the incredible insight in Ferguson [6] who studied the general dependent sequence of X_i in odds problem.

¹This is an abbreviation of the original version.

Theorem 1 (Ano, Kakie and Miyoshii [3]) Assume that $0 < \alpha_i, \beta_i < 1$ for each $i \in \mathcal{N}$. The optimal single selecting strategy for the non-homogeneous Markov-dependent trials is given by

$$\tau^* = \min \left\{ i \in \mathcal{N} : X_i = 1 \text{ \& } \sum_{j=i+1}^N r_{ij} < 1 \right\} = \min \{ i \geq i^* : X_i = 1 \}.$$

Assume that $X_1 = 1$ a.s., then the probability of win holds the inequality

$$P_N(\text{win}) = P_N(\alpha_0, \dots, \alpha_{N-1}, \beta_0, \dots, \beta_{N-1}) \geq R_{i^*-1} V_{i^*-1},$$

where $R_s = \sum_{j=s+1}^N r_{sj}$ and $V_s = \alpha_s \prod_{k=s+1}^{N-1} (1 - \beta_k)$.

Next theorem is the main result of this paper.

Theorem 2 Assume that $X_1 = 1$, a.s. If $R_s = \sum_{j=s+1}^N r_{sj}$ with $s = i^* - 1$, then

- (i) $P_N(\text{win}) \geq R_s V_s > R_s e^{-R_s}$.
- (ii) If $R_s = R_{s(N)} \rightarrow 1$ as $N \rightarrow \infty$, then $\lim_{N \rightarrow \infty} P_N(\text{win}) > 1/e$.

Proof.

- (i) By the optimality equation $M_i = \max \left\{ V_i, \sum_{j=i+1}^N \mathbf{P}_{ij} M_j \right\}$ and since $\sum_{j=i+1}^N \mathbf{P}_{ij} M_j$ is decreasing in i , we have

$$\begin{aligned} P_N(\text{win}) &= P_N(\text{win} | X_1 = 1) \\ &=: M_1 = \max \left\{ V_1, \sum_{j=2}^N \mathbf{P}_{1j} M_j \right\} \\ &\geq \sum_{j=2}^N \mathbf{P}_{1j} M_j \geq \sum_{j=2}^N \mathbf{P}_{1j} V_j, \end{aligned} \tag{2}$$

where $V_i = P_N(\text{win by stop at } X_i = 1 | X_1 = 1) = \alpha_i \prod_{j=i+1}^{N-1} (1 - \beta_j)$. Using $r_{ij} = (1 - \alpha_i) \alpha_{i+1} / \alpha_i (1 - \beta_{i+1})$ for $j = i + 1$; $= \alpha_{j-1} \beta_j / (1 - \beta_{j-1}) (1 - \beta_j)$ for $j > i + 1$, we have

$$P_N(\text{win}) \geq \sum_{j=2}^N \frac{\mathbf{P}_{1j} V_j}{V_1} V_1 = \sum_{j=2}^N r_{1j} V_j = R_1 V_1.$$

- (ii) Note that $V_s = \prod_{k=s+1}^{N-1} q_{sk} / (\prod_{k=s+1}^{\tilde{N}-1} (1 - \beta_k))$, where $\tilde{N} = N$ if N is an even integer, and $\tilde{N} = N - 1$ if N is an odd integer. Since $1 - \beta_k < 1$,

$$P_N(\text{win}) \geq R_s V_s = \frac{R_s \prod_{k=s+1}^{N-1} q_{sk}}{\prod_{k=s+1}^{\tilde{N}-1} (1 - \beta_k)} > R_s \prod_{k=s+1}^{N-1} q_{sk}.$$

From $R_s = \sum_{k=s+1}^N (1/q_{sk} - 1)$, we have $\sum_{k=s+1}^N (1/q_{sk}) = R_s + N - s$. By the inequality for arithmetic mean and geometric mean, then

$$\left(\prod_{k=s+1}^N \frac{1}{q_{sk}} \right)^{\frac{1}{N-s}} = \left(\frac{1}{\prod_{k=s+1}^N q_{sk}} \right)^{\frac{1}{N-s}} \leq \frac{\sum_{k=s+1}^N \frac{1}{q_{sk}}}{N-s} = 1 + \frac{R_s}{N-s}$$

and thus $\prod_{k=s+1}^N q_{sk} \geq (1 + R_s/(N-s))^{-(N-s)}$. From $(1 + R_s/(N-s))^{-(N-s)} \downarrow e^{-R_s}$ as $N \rightarrow \infty$, it follows that

$$P_N(\text{win}) > R_s \prod_{k=s+1}^{N-1} q_{sk} \geq R_s \left(1 + \frac{R_s}{N-s} \right)^{-(N-s)} > R_s e^{-R_s} \rightarrow 1/e,$$

as $N \rightarrow \infty$.

□

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